Brownian motion and Stochastic Calculus
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Assignment 1

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$.

On hitting times of closed and open sets

Let E be a finite-dimensional Euclidean space, and let X be an E-valued and $\mathbb{F}$-adapted process.

1) Prove that if $G \subset E$ is open and X has right- or left-continuous trajectories, then $\rho_{X,G}$ and $\tau_{X,G}$ are $\mathbb{F}$–optional times (see Definition 1.3.3 in the lecture notes)

2) Show that if $G \subset E$ is closed and X has continuous trajectories, then $\tau_{X,G}$ is an $\mathbb{F}$–stopping time.

3) Show that if $G \subset E$ is closed, and X is càdlàg then $\theta_{X,G}$ is an $\mathbb{F}$–stopping time (see Definition 1.3.5 in the lecture notes).

Properties of stopping times

1) Show that if τ is an $\mathbb{F}$–optional time, and $\theta \in (0, +\infty)$, then $\tau + \theta$ is an $\mathbb{F}$–stopping time.

2) Show that if τ and ρ are $\mathbb{F}$–stopping times, so are $\tau \wedge \rho$, $\tau \vee \rho$ and $\tau + \rho$.

3) Show that if τ and ρ are $\mathbb{F}$–optional times, then $\tau + \rho$ is also an $\mathbb{F}$–optional time. It is moreover an $\mathbb{F}$–stopping time if either τ and ρ are positive, or if $\tau > 0$ and τ is an $\mathbb{F}$–stopping time.

4) Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{F}$–optional times. Show that the following four random times are $\mathbb{F}$–optional times
$$\sup_{n\in\mathbb{N}} \tau_n,\ \inf_{n\in\mathbb{N}} \tau_n,\ \overline{\lim_{n\to+\infty}}\, \tau_n,\ \underline{\lim_{n\to+\infty}}\, \tau_n.$$
Furthermore, if the $(\tau_n)_{n\in\mathbb{N}}$ are actually $\mathbb{F}$–stopping times, show that $\sup_{n\in\mathbb{N}} \tau_n$ is an $\mathbb{F}$–stopping time too.

Hitting times and completeness of $\mathbb{F}$

Let X be an $\mathbb{R}$-valued and right-continuous process. The goal of this exercise is to show that for any $M \in \mathbb{R}$, the hitting time $\tau_{X,[M,+\infty)}$ is an $\mathbb{F}$–stopping time when $\mathbb{F}$ is $\mathbb{P}$-complete.

1) Given any $\mathbb{F}$–stopping time σ which is below $\tau_{X,[M,+\infty)}$, define
$$\sigma^+ := \inf\left\{ t \geq \sigma \colon \sup_{\sigma \leq u \leq t} X_u \geq M \right\}.$$
Show that $\sigma \leq \sigma^+ \leq \tau_{X,[M,+\infty)}$, that σ^+ is still an $\mathbb{F}$–stopping time, and that σ^+ is strictly greater than σ whenever $\sigma < \tau_{X,[M,+\infty)}$.

2) Let $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ consist of the set of all $\mathbb{F}$–stopping times σ satisfying $\sigma \leq \tau_{X,[M,+\infty)}$, and define
$$\sigma_\infty := \operatorname{essup}^{\mathbb{P}} \mathcal{T}_{0,\tau_{X,[M,+\infty)}}.$$
Prove that the family $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ is upward directed and deduce that there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ valued in $\mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ such that
$$\sigma_\infty = \sup_{n\in\mathbb{N}} \tau_n.$$

3) Show that $\sigma_\infty \in \mathcal{T}_{0,\tau_{X,[M,+\infty)}}$ and that $\sigma_\infty^+ = \sigma_\infty$, $\mathbb{P}$–a.s.

4) Deduce that $\sigma_\infty = \tau_{X,[M,+\infty)}$ and conclude.